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MODEL OF THE PENETRATION OF AN UPPER UNIFORM LAYER INTO A STRATIFIED FLUID

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We study an integral model of the penetration of a uniform layer of fluid under the action of a tangential stress applied to the surface. The conservation equations for mass, momentum, and energy are closed by the penetration law of the nonmoving fluid into the upper uniform layer. An important feature of our model is that the nonuniformity of the velocity field due to the presence of "free" vortices in the flow is taken into account.

Two penetration regimes are identified: a subcritical regime, where the penetration of the fluid into the layer occurs because of externally induced turbulence of the uniform layer, and a supercritical regime in which turbulence at the surface is transported by large-scale vortices generated by a flow instability with a velocity shear. It is shown that for an initial bilayered density distribution, and also in the case of a continuous density distribution following a power law, there exist singular solutions of the system of equations corresponding to the supercritical penetration regime, and these solutions determine the asymptotic behavior at large times. These solutions are characterized by the constancy of the global Richardson number Ri_u, calculated with respect to the mean values of the buoyancy and velocity of the upper layer. Hence the hypothesis Ri_u = const used in several models [1] to close the momentum equation is correct asymptotically in the framework of our model. Inclusion of the lateral friction for flow in a channel of finite width destroys the asymptotic form of the penetration and the solution is transformed into the subcritical regime. Comparison with experimental results in circular troughs shows that our model gives a satisfactory description of the supercritical penetration for a bilayer [2] and for a continuous initial density distribution [3].

The process of mixing in the flow of a stably stratified fluid is a complex and important problem. Transport of momentum and heat from the surface into the bulk of the ocean determines the formation and time behavior of the upper thermocline. The transport mechanism is related to the development of instabilities in the shear flow and to turbulent exchange between layers of different densities. An adequate mathematical description of the formation and structure of the upper layer of the ocean is possible only with the use of turbulent models [4]. However, for a certain class of flows a simple integral model can be used which gives the time behavior of the average quantities, which completely characterize this class of flow.

In experiments and in observations it is noted that a stress applied to the surface of a stratified fluid at rest leads to a well-mixed layer with a nearly constant velocity and density and the layer is separated from the unperturbed nonmoving fluid by a thin transition layer where there are large gradients. In an idealized formulation of the problem, one assumes that the layer is uniform and has density $\rho(t)$, and a horizontal component of the velocity $u(t)$ (the only component which is nonzero), and the small-scale motion extends to a depth $h(t)$ with intensity $q(t)$ (Fig. 1, region I). Below the line $y = -h(t)$ there is the nonmoving

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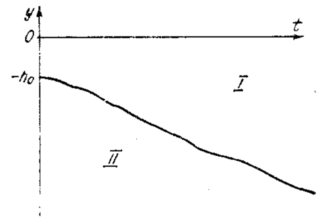

Fig. 1

stratified fluid [$u^0 = 0$, $q^0 = 0$, $\rho = \rho^0(y)$, $d\rho^0/dy < 0$, region II]. A given stress $\tau = \tau^*(t)$ is applied to the surface layer ($y = 0$). The mass flux through the surface is zero. This model describes the process of penetration of the uniform turbulent layer under the action of a surface stress (e.g., wind).

The integral model has been applied with success by many authors to describe the dynamics of the upper layer of the ocean [5, 6]. Besides its simplicity, this approach is attractive because in deriving the dynamical equations of the uniform layer one can do without the closure hypothesis, which is an integral part of the turbulent mixing models. However, in order to close the conservation equations for mass, momentum, and energy, it is necessary to specify a law for the penetration of the nonmoving fluid into the uniform layer.

The following mixing law is widely used in oceanography and meteorology:

$$dh/dt = Bu^*/\mathrm{Ri}, \tag{1}$$

where $\mathrm{Ri} = (\rho^0(-h) - \rho)gh/\rho_0\tau^*$; $u^* = \sqrt{\tau^*}$ is the frictional velocity, g is the acceleration of gravity, ρ_0 is the initial density of the fluid at the surface. The dependence (1) is based on the results of laboratory experiments in circular troughs with linear initial stratification ($B \cong 2.5$) [3], and also similar experiments [2] for bilayer stratification. More recent analysis of the experiments in circular troughs has shown [1] that good results are obtained when the equations are closed by the relation

$$\mathrm{Ri}_u = (\rho^0(-h) - \rho)gh/\rho_0 u^2 \equiv \mathrm{const}. \tag{2}$$

Here Ri_u is the global Richardson number, calculated with respect to the mean values of the buoyancy and velocity of the upper layer. The dependence (2) leads to a different penetration law than in (1),

$$dh/dt = n\,\mathrm{Ri}_u^{1/2}\,\mathrm{Ri}^{-1/2}u^* \tag{3}$$

with $n = 1$ for a bilayer stratification and $n = \frac{1}{2}$ for linear stratification. In [1] it was assumed that $\mathrm{Ri}_u \sim 0.6$. However, in the experiments of [7] performed in a circular trough, it was shown that the dependence (2) is not satisfied in all cases and Ri_u can become significantly larger than unity.

The model presented below is based on the conservation laws of mass, momentum, and energy and reflects the fact that in the supercritical regime the nonuniformity of the velocity q is coupled not only with the turbulence induced in the flow because of the external stress, but also on account of internal instabilities of the shear flow. The penetration law of the layer is determined by the intensity of small-scale motion and is written in the form

$$dh/dt = Aq/\mathrm{Ri}. \tag{4}$$

The model considered here combines the postulates of the model of [3] with the conclusions of [1]. The essence of the model is as follows. In the absence of lateral friction there is a class of initial profiles $\rho^0(y)$ (including the bilayered [2] and linear stratification [3] cases) in which there exist singular solutions of the system of conservation equations for mass, momentum, and energy plus Eq. (4). These solutions are characterized by the property $\mathrm{Ri}_u = \mathrm{const}$ and are the asymptotic limit ($t \to \infty$) of other solutions of the system. For flow in channels of finite width the friction at the lateral boundaries leads to a change in the asymptotic behavior of the solutions.

Let the uniform layer have density ρ^+, velocity u^+, and depth h_0 at $t = 0$. For simplicity we assume that the buoyancy $b^0(y) = (\rho^- - \rho_0)g/\rho_0$ in the lower layer is distributed according to a power law ($\nu \equiv \mathrm{const} > 0$)

$$b^0(y) = b^- + \nu(-y)^\gamma, \; y < -h_0. \tag{5}$$

The dependence (5) includes two important special cases: a bilayered liquid ($\nu = 0$, $b^- > b^+$) and linear stratification ($\gamma = 1$, $h_0 = 0$).

The dynamical equations of the uniform layer in the Boussinesque approximation ($|\rho - \rho_0| \ll \rho_0$) are obtained from the conservation laws of momentum, energy, and mass after integration with respect to y from zero to a certain value y_0, where $y_0 < -h(t)$. On the boundaries $y = 0$ and $y = y_0$ the Reynolds stress and the mass flux are assumed to be given:

$$\tau^* = -\overline{u'v'}\,|_{y=0},\; \overline{\rho'v'}\,|_{y=0} = \overline{\rho'v'}\,|_{y=y_0} = \overline{u'v'}\,|_{y=y_0} = 0.$$

Here u', v', ρ' are the fluctuating components of the velocity vector and density. The integrated equations of motion have the form [5]

$$\begin{gathered} dhu/dt = \tau^*, \\ \frac{d}{dt}\left((1/2)\,u^2h + (1/2)\,q^2h + \int\limits_{y_0}^{0} by\,dy\right) = \tau^*u - \varepsilon h, \\ \frac{d}{dt}\left(\int\limits_{y_0}^{0} b\,dy\right) = 0, \end{gathered} \tag{6}$$

where u(t) is the mean horizontal velocity

$$q(t) = \left(\frac{1}{h}\int\limits_{-h}^{0} (u^2(t,y) + v^2(t,y) + w^2(t,y))\,dy - u^2\right)^{1/2}$$

and characterizes the nonuniformity of the velocity field due to small-scale motion. Terms including correlations of the pressure and velocity and also averages of triple products of the fluctuating components of the velocity at $y = 0$ are assumed to be small and are omitted in the energy equation. The quantity ε describes energy dissipation. Because q measured the nonuniformity of the flow due to fluctuating motion of all scales, one can put $\varepsilon = 0$ if we ignore thermal dissipation of energy.

The equations (6) are closed by the penetration law (4) which has a simple interpretation: the rate of increase of the potential energy V on account of mixing of the fluids is proportional to the energy of the vortices formed under the action of the surface stress

$$dV/dt = (1/2)(b^0(-h) - b)h(dh/dt) = (1/2)A\tau^*q. \tag{7}$$

The relation (7) differs from the expression for the rate of change of the potential energy used in [3] in the derivation of the law (1) by the fact that the velocity of vertical transport of vortices is determined not only by the friction velocity u^*, but also by the mean-square fluctuation velocity q generated by the internal shear instability.

With the help of (5), system (4) and (6) can be rewritten in the form

$$dh/dt = \sigma q,\; h\,dh/dt = \tau^* - \sigma qu,\; hq\,dq/dt = (\sigma q/2)\;(u^2 - q^2 - c^2), \tag{8}$$

where $c^2 = (b^- - b^+)h_0 + (\nu/(\gamma+1))/h^{\gamma+1}_0 + (\gamma\nu/(\gamma+1))h^{\gamma+1}$; $\sigma = A/\mathrm{Ri}$; $\mathrm{Ri} = \tau^*/c^2$. The density of the layer ρ is eliminated from the equations by integration of the conservation of mass law. The quantity q^2, characterizing the nonuniformity of the velocity field, is composed of the energy of the vortices $q_i{}^2$ generated within the flow, and the kinetic energy $q_e{}^2$ of vortices generated on the upper boundary. The value of q_e is proportional to u^*, i.e., $q_e = au^*$, $a = a(h)$, and $a \equiv a_0$ if one does not take into account the scattering of the vortex energy with increasing depth of the mixed layer. Therefore the energy equation describes the generation of the energy of "free vortices" $q_i{}^2$ in the shear flow. If q_i goes to zero and the derivative dq/dt is negative, then the increase in kinetic energy on account of the internal redistribution of the flow is completely cancelled by the change of potential energy and "free vortices" do not exist in the flow. In this case the energy equation can be replaced by the relation $q \equiv q_e$. Because the quantity q_e is much less than q_i in the supercritical flow case ($dq/dt > 0$) the rate of penetration of the uniform layer significantly depends on which flow regime is realized.

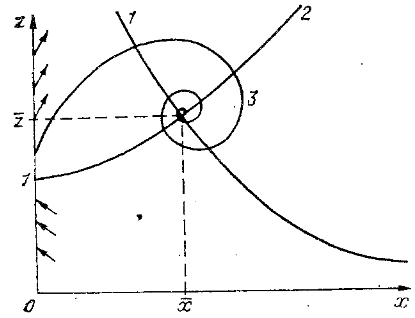

Fig. 2

We consider the bilayered and continuous initial stratification cases separately.

Bilayer Model. Let $\nu = 0$. In this case, $c = \text{const}$, and if the depth of the channel $H \to \infty$, then c is the propagation velocity of long internal waves. In the (x, z) plane ($x = q/c$, $z = u/c$) the trajectories of the system (8) are coincident with the integral curves of the equation

$$dz/dx = [2(1 - Axz)]/[A(z^2 - x^2 - 1)]. \tag{9}$$

The stationary point of (9) is found from the intersection of the curves $Axz = 1$ and $z^2 = x^2 + 1$ (Fig. 2, curves 1 and 2, respectively).

In the region $x > 0$, $z > 0$, there exists a unique singular point $(\bar{x}, \bar{z})$ of Eq. (9) which is a stable focus. When $\tau^* = \text{const}$ the stationary point $(\bar{x}, \bar{z})$ determines the equilibrium penetration law

$$dh/dt = A\bar{q}/\mathrm{Ri},\ u = \bar{u},\ q = \bar{q} \tag{10}$$

with $\mathrm{Ri}_u \equiv \text{const}$. Any trajectory in the neighborhood of the singular solution (10) is drawn in toward it (Fig. 2, curve 3) so that flow with $\mathrm{Ri}_u \equiv \text{const}$ is realized in the limit $t \to \infty$.

If the stress τ^* is applied to a nonmoving bilayered fluid ($u^0 = 0$, $q^0 = 0$) at $t = 0$ then one can identify three phases in the solution for the penetration of the uniform layer: an acceleration of the upper layer with $q \equiv q_e$, $u^2 < q^2 + c^2$, a nonmonotonic increase of the velocity and a transition into the asymptotic regime of equilibrium penetration. Of particular interest is the middle phase of the motion and it can be shown that even in the absence of lateral friction the velocity in the upper layer varies nonmonotonically. It is seen from Fig. 2 that flow with $\mathrm{Ri}_u < 1$ is realized asymptotically and the quantity $\mathrm{Ri}_u|_{t\to\infty} \to \mathrm{Ri}_{\bar{u}}$ weakly depends on the choice of the constant A if $1 \leqq A \leqq \infty$. Indeed,

$$\mathrm{Ri}_{\bar{u}} = 2\Big/\left(1 + \sqrt{1 + 4A^{-2}}\right). \tag{11}$$

When $A = 1$, $\mathrm{Ri}_{\bar{u}} = 2/(1 + \sqrt{5}) \cong 0.618$ which is consistent with the use of this value from the experimental data in [1]. It is shown below that the value $A = 1$ also corresponds to the experimental results for the case of linear initial stratification.

Linear Stratification. Let $\gamma = 1$, $h_0 = 0$, i.e., $c^2 = c^2(h) = \frac{1}{2}\nu h^2$, $\mathrm{Ri} = \tau^*/c^2$, $\sigma = A\mathrm{Ri}^{-1}$. The system (8) is homogeneous and reduces to an autonomous system. In the (x, z) plane ($x = q/c$, $z = u/c$) the trajectories of the system (8) are coincident with the integral curves of the equation

$$dz/dx = [2(1 - 2Axz)]/[A(z^2 - 3x^2 - 1)]. \tag{12}$$

The singular point of (12) is determined from the intersection of the curves $2Axz = 1$ and $z^2 = 3x^2 + 1$. As in the case of bilayer stratification, in the region $x > 0$, $z > 0$ there exists a unique stationary point $(\bar{x}, \bar{z})$ which is a stable focus. The stationary point determines the singular solution of the system (8)

$$h = \bar{h}t^{1/2},\ u = \bar{u}t^{1/2},\ q = \bar{q}t^{1/2}, \tag{13}$$

where $\bar{h} = (\sqrt{2}\tau^*/\sqrt{\nu}\,\bar{y})^{1/2}$; $\bar{c} = \sqrt{\nu/2}\,\bar{h}$. The solution (13) is an exact solution for the penetration of a uniform upper layer ($\tau^* \equiv \text{const}$) into an initially nonmoving ($u^0 = 0$, $q^0 = 0$) linearly stratified fluid; it is characterized by the relation

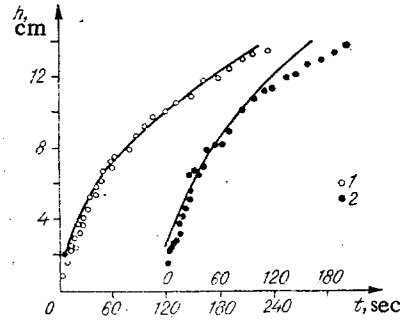

Fig. 3

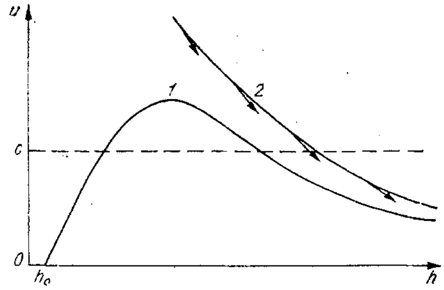

Fig. 4

$$\mathrm{Ri}_u = c^2/u^2 = \mathrm{const}.$$

Hence the flow realized in [3] is described by the singular solution (13). In Fig. 3, experimental data is shown for the thickness of the uniform layer as a function of time, obtained in [3] for $\nu = 1.98\ \mathrm{sec}^{-2}$, $\tau^* = 0.995\ \mathrm{cm}^2/\mathrm{sec}^2$ (points 1) and $\nu = 3.84\ \mathrm{sec}^{-2}$, $\tau^* = 2.12\ \mathrm{cm}^2/\mathrm{sec}^2$ (points 2). The solid curves correspond to the exact solution (13). In the portions of the curves where the effect of friction at the lateral boundaries of the trough can be neglected, the experimental dependence is described by the solution (13).

The effect of the constant A on the singular solution (13) is insignificant when $1 \leqq A \leqq \infty$, as in the case of a bilayered fluid ($\nu = 0$). In the solution (13) the quantity Ri_u can be expressed in terms of A, as follows:

$$\mathrm{Ri}_u = 2/(\sqrt{1 + 3A^{-2}} + 1).$$

For A = 1, $\mathrm{Ri}_u = 2/3$ and differs only slightly from the limiting value $\mathrm{Ri}_u \cong 0.618$ for $\nu = 0$. The dependence of the function h = h(t) [in (13)] on A is even weaker because

$$h = [(2/\nu)\mathrm{Ri}_u]^{1/4}(\tau^* t)^{1/2}$$

and $2/3 \leqq \mathrm{Ri}_u < 1$ when $1 \leqq A < \infty$. Therefore, system (8) with the value A = 1 gives a satisfactory description of the penetration of a uniform layer for both linear and bilayered initial stratification.

Note. A continuous initial stratification ($h_0 = 0$) with an arbitrary power law dependence (5) of the density on depth is treated in a similar way. In this case the system (8) reduces to an autonomous system in the (x, z) plane, where x = q/c, z = u/c, and Ri_u = const for the singular solution of (8).

For a channel of finite width it is necessary to take into account friction at the lateral walls. The system (8) in this case describes the time behavior of the uniform layer if the surface stress τ^* is replaced by an "effective" stress $\tau = \tau^* - \tau_w$, where $\tau_w = c_w h u^2 L^{-1}$, c_w is the coefficient of friction, and L is the width of the channel [1].

Let $c_w \equiv \mathrm{const} \neq 0$. The asymptotic property of the solutions of (8) discussed above is now not correct, because with increasing h and u the quantity τ_w becomes comparable to τ^*. In studying the behavior of the trajectories (8) it is convenient to consider their projections onto the (h, u) plane. A trajectory h = h(t), u = u(t) (Fig. 4, curve 1) cannot intersect the line $\tau = \tau^* - c_w L^{-1} h u^2 = 0$ (curve 2) because the trajectories of the system (8) lie outside of this curve in the region $\tau > 0$. Therefore, when $t \to \infty$, $u(t) \to 0$ and the solution must become subcritical ($u^2 < q_e^2 + c^2$) and this leads to a significant change in the penetration rate of the layer.

In the experimental results for a bilayer in a circular trough [2] the effect of the lateral friction is much stronger than in the experiments of [3] (see Fig. 3). When $\mathrm{Ri} \leqq 100$ (in [2]) the acceleration phase of the upper layer was realized and the system passed into the supercritical penetration regime. When $\mathrm{Ri} \geqq 500$, the flow remained subcritical the entire time which led to a sharp decrease in the penetration rate of the uniform layer. The hypothesis that $q = q_e = \alpha_0 u^*$ for subcritical flow corresponds more to the plane-parallel case. In an annular trough the nonuniformity of the velocity field is due not only to turbulence induced in the fluid by the moving baffle but also by the presence of radial motion [7]. Therefore, in order to correctly describe subscritical flow in an annular trough, more detailed information is necessary on the structure of the flow. In the supercritical case the flow geometry is less important because the nonuniformity of the flow due to the generation of large vortices in the shear layer dominates over nonuniformities arising from other mechanisms.

The hypothesis that the average velocity and density profiles are uniform along the vertical in the upper layer is a rather crude approximation. In actual fact, the well-mixed layer is separated from the nonmoving fluid by a layer whose thickness can be a significant fraction of that of the upper layer. Nevertheless, the model considered here gives not only qualitative agreement with observation, but also quantitative agreement. Apparently this can be explained by the fact that the transition layer is dynamically neutral, i.e., in this layer the kinetic energy of the fluctuating motion produced by the redistribution of the flow is dissipated in overcoming buoyancy forces. Therefore the quantity q^2 in (8) is the difference of the kinetic and potential energies of the flow due to the nonuniformity of the flow, i.e., the "free" energy of the vortices, and is not cancelled by the increase in the buoyancy. Hence (8) describes formally the the time behavior of the upper layer with a transition zone if h is interpreted as the distance from the surface of the fluid to the middle of the transition layer. The inclusion of dissipation parametrized by an expression of the form $\varepsilon = kq^3h^{-1}$ does not change the qualitative behavior of the solutions of the system (8).

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